Film-penetration Model for Mass and Heat Transfer

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The two major models of the mechanism of mass and heat transfer between two phases are the film theory (*12*) and the penetration theory (*2*, *3*). The film theory assumes that there is a region in which steady state molecular transfer is controlling; the penetration theory assumes that the interface is continuously replaced by eddies and that unsteady state molecular transfer into the eddies controls the transfer in this region.

There are three classes of problems to which these theories have been applied: (1) transfer between a solid and a fluid in turbulent flow; (2) transfer between two fluids, at least one of which is in turbulent flow; and, (3) transfer between two fluids in an apparatus of discontinuous geometry such as packed column. In flow in a packed column the concept of surface renewal by eddies is replaced by that of a laminar liquid which mixes at discontinuities in the packing (*2*). If the flow is laminar except at the points of mixing, neither the film theory nor the analysis developed below will apply to the liquid phase.

In the absence of any other resistances the film theory predicts a first-power dependence of the transfer rate on the diffusivity or thermal conductivity, and the penetration theory predicts a square root dependence. Danckwerts (*3*) has shown that neither the film nor penetration theory is completely valid for a packed column, and Hanratty (*5*) points out that at high Schmidt numbers the penetration theory gives better results than the film theory.

The purpose of this paper is to show that the film and penetration theories are not separate, unrelated concepts but rather are limiting cases of a more general model and that the two theories, rather than being mutually exclusive, are actually complementary.

THEORETICAL DEVELOPMENT

The boundary layer extending from the front end of a flat plate or from the inlet to a conduit will be considered. A laminar zone is assumed to exist which is bounded by a turbulent region, and heat or mass is being transferred between the surface and the fluid. For a short distance along the plate the transfer must be by an unsteady state mechanism, for the penetration of the diffusing substance will take a finite contact time (corresponding to a finite distance along the surface) to reach the edge of the film. Some distance along the surface the transfer will reach its steady value and there will be no more accumulation in the film. The short distance corresponds to the penetration theory, the long distance to the film theory, and for intermediate distances the transfer process has the characteristics of both mechanisms.

The transfer between a gas and stirred liquid, which as postulated by Danckwerts has its surface randomly replaced by eddies of fresh fluid from the bulk of the liquid, will be considered. If the eddies remain in the surface a short period of time, each element may be assumed to absorb matter or heat at the interface by unsteady state conduction. As the life of an element increases, the penetration into the element increases and again, after a long enough time, a steady gradient will be set up in the element, no more accumulation will take place, and material will be transferred through the element. Thus the old elements obey the film theory, the young ones the penetration theory, and the middle aged ones have characteristics of both mechanisms. In this intermediate case the penetration has reached the inner side of the element but the steady gradient has not yet been attained. If elements of all ages are present, all three types of transfer take place simultaneously and the model which includes all the cases might be called the *film-penetration* model.

To put this discussion on a quantitative basis it is necessary to modify the Danckwerts-Higbie theory as follows:

The transfer equation is

$$\frac{\partial C}{\partial \theta} = D\,\frac{\partial^2 C}{\partial y^2} \qquad (1)$$

This neglects all velocity gradients.

The boundary conditions are

$$\theta = 0, \quad C = C_L \qquad (2)$$

$$y = 0, \quad C = C_i \qquad (3)$$

$$y = L, \quad C = C_L \qquad (4)$$

C_i and C_L are assumed independent of θ. The difference between this model and the penetration model is due to Equation (4). It is assumed here that at some distance L below the surface the concentration remains constant at C_L and that a freshly formed surface has this concentration. This condition allows the transfer into an old element to approach the steady state value although the penetration model, which makes L infinite, excludes this limit.

The assumptions have a number of interpretations. At the free surface of a well-stirred liquid in contact with a gas the authors assume, with Danckwerts, that eddies penetrate the surface from the bulk of the liquid and in so doing displace older surface elements, which return to the liquid bulk. The bulk concentration is held constant in time by using, say, a continuous-flow stirred tank.

Below the surface the liquid is well mixed and if the average thickness of the surface elements can be considered to be L, then the concentration at $y = L$ is held constant in time at C_L, the bulk concentration, by the high rate of turbulent transfer from this point to the bulk. That is, the eddies enter the surface with the concentration C_L, since they come from the turbulent region, and when the penetration into any element reaches L, the diffusing material is immediately carried into the bulk by turbulent transfer. In a similar manner the concentration at $y = 0$ is held constant by the high rate of transfer from the gas to the interface. If we admit that the turbulent transfer coefficient is not infinite, so that there is a concentration gradient below L, then eddies may be visualized as entering the surface with the average concentration over the region bounded by the planes $y = L$ and $y = 2L$, and this concentration approaches C_L as the turbulent coefficient increases. It is also necessary that the amplitude of the concentration oscillations at L during the life of a surface element be much less than $C_i - C_L$.

This analysis implies that surface elements differ from other fluid elements in that they may be bombarded on their lower face by a number of eddies before they are displaced from the surface and that because of surface tension an element in the surface is in a stable position and may be removed only by an extra large "kick" from below. The same analysis may also be applied with little change to the fluid-solid interface in a continuous-flow stirred tank, but if the fluid bulk has a mean velocity some modifications are necessary.

At the interface between a solid and a moving turbulent fluid, as in steady flow in a conduit, it is well known that the fluid is in a state different from that of the outer turbulent regions, for the eddying motion is damped by the presence of the wall. Consequently, it is convenient to visualize the fluid adjacent to the wall as a laminar film of average thickness L which is mixed at various intervals by disturbances from the surrounding fluid. The mixing is presumed to occur over a short axial distance and to be violent enough, not only to eliminate the concentration gradient which has built up in the film but to remove the fluid bodily and replace it by fresh fluid from the region $y > L$. If the fluid is highly turbulent in this outer region, then the concentration in the film after mixing is C_L, which now is the bulk concentration, and the concentration at L will always be held at C_L by the turbulent transfer from the plane L to the bulk. In order for C_L to be independent of the age of an element, the mixing must be frequent enough so that the bulk concentration does not change appreciably in the distance between two successive mixings. (This qualification also applies to the penetration model and will be examined later.)

Since the transfer coefficient from $y = L$ to the liquid bulk is not infinite, the assumption that the concentrations at $\theta = 0$ and $y = L$ are the same [Equations (2) and (4)] necessitates, as in the previous case, that the concentration gradient from $y = 0$ to $y = L$ be much steeper than the gradient from $y = L$ to $y = 2L$. In addition, it is now necessary that the concentration in the region $y = L$ to $y = 2L$ does not change appreciably over the distance between two successive mixings and that between mixings the amplitude of the concentration fluctuations at L is small.

The first of these examples has been approached from the penetration viewpoint, the second from the film viewpoint, but the assumptions which have been made make the two cases identical. There is a strong implication in the previous discussion that there is a velocity gradient in the film in the case of conduit flow, but for the sake of simplicity this gradient is neglected.

The solutions of Equations (1) to (4) are available (*1*) and from the solutions the instantaneous transfer rates are found to be, in the nomenclature of mass transfer,

short times

$$N = \Delta C \sqrt{\frac{D}{\pi\theta}} \cdot \left[1 + 2 \sum_{n=1}^{\infty} \exp\left\{-\frac{n^2L^2}{D\theta}\right\}\right] \quad (5a)$$

long times

$$N = \Delta C \frac{D}{L} \cdot \left[1 + 2 \sum_{n=1}^{\infty} \exp\left\{-n^2\pi^2 \frac{D\theta}{L^2}\right\}\right] \quad (5b)$$

These equations are equivalent, but the first converges rapidly for short times and the second for long times. The equations show clearly that for short times the penetration theory is approached,

$$N = \Delta C \sqrt{\frac{D}{\pi\theta}} \quad (6)$$

and for long times the film theory is approached,

$$N = \Delta C \frac{D}{L} \quad (7)$$

It can now be seen that what is meant by short times is that the group $(\theta D)/(L^2)$ is small.

The penetration theory holds for the region $\theta \ll (L^2/D)$ and the film theory is valid for the region $\theta \gg L^2/D$. The comparison can be seen in Figure 1, which shows clearly that the film and penetration theories are asymptotes of the film-penetration model.

In general, the point rates given above are of less interest than the average rates, and for transfer across a surface consisting of a multiplicity of surface ages the mean rate is

$$\bar{N} = \frac{1}{A'} \int_0^{A'} N \, dA \quad (8)$$

or in terms of a surface-age distribution function,

$$\bar{N} = \int_0^{\infty} N\psi(\theta) \, d\theta \quad (9)$$

If the lives of all elements are the same, the Higbie distribution function applies (*3*),

$$\psi(\theta) = \frac{1}{\theta'} \quad \theta < \theta' \qquad \psi(\theta) = 0 \quad \theta > \theta' \quad (10)$$

and Equation (5) with (9) and (10) gives short time

$$\bar{N} = \Delta C 2 \sqrt{\frac{D}{\pi\theta'}} \left[1 + 2\sqrt{\pi} \cdot \sum_{n=1}^{\infty} ierfc \frac{nL}{\sqrt{D\theta'}}\right] \quad (11a)$$

long time

$$\bar{N} = \Delta C \frac{D}{L} \left[1 + \frac{2}{\pi^2} \frac{L^2}{D\theta'} \left(\frac{\pi^2}{6} - \sum_{n=1}^{\infty} \exp\left\{-n^2\pi^2 \frac{D\theta'}{L^2}\right\}\right)\right] \quad (11b)$$

ierfc is the integral of the complementary error function and is tabulated (*1*).

The Danckwerts distribution function assumes that the elements of surface are replaced in a random fashion, so that (*2*)

$$\psi(\theta) = Se^{-S\theta} \quad (12)$$

If this is used in place of Equation (10) the averaging process yields

short average life (rapid replacement)

$$\bar{N} = \Delta C \sqrt{DS} \left[1 + 2 \cdot \sum_{n=1}^{\infty} \exp\left\{-2nL\sqrt{\frac{S}{D}}\right\}\right] \quad (13a)$$

long average life (slow replacement)

$$\bar{N} = \Delta C \frac{D}{L} \left[1 + 2 \sum_{n=1}^{\infty} \frac{1}{1 + n^2\pi^2 \frac{D}{SL^2}}\right] \quad (13b)$$

Equation (11) reduces to the penetration theory with the Higbie distribution for short times ($\theta' D/L^2$ small),

$$\bar{N} = 2\Delta C \sqrt{\frac{D}{\pi\theta'}} \quad (14)$$

and to the film theory for long times. In the latter limit the point rate and average rate are the same, and so this asymptote is given by Equation (7). At the former limit all the elements of surface are too young for the penetration to reach L, and at the latter limit most of the surface is old enough to have been completely penetrated.

Equation (13) contains instead of the contact time the variable S, which is the fraction of the surface replaced by fresh fluid per unit time. When the surface is replaced at a rapid rate (SL^2/D large), most of the surface is too young to have been penetrated and Equation (13) reduces to Danckwerts's equation

$$\bar{N} = \Delta C \sqrt{DS} \quad (15)$$

When $(SL^2)/D$ is small, the major part of the surface has been completely penetrated and Equation (13) reduces to Equation (7), the surface having the characteristics of a film.

Equations (11) and (13) converge rapidly and only a small error is made by dropping higher order terms in Equations (11*a*), (11*b*), and (13*a*) and neglecting the constant in the denominator of the series in Equation (13*b*). The equations then reduce to

Higbie Distribution

young, $Z' < 1.0$

$$\bar{N} = 2\Delta C \sqrt{\frac{D}{\pi\theta'}} \cdot \left[1 + 2\sqrt{\pi} ierfc \frac{L}{\sqrt{D\theta'}}\right] \quad (16a)$$

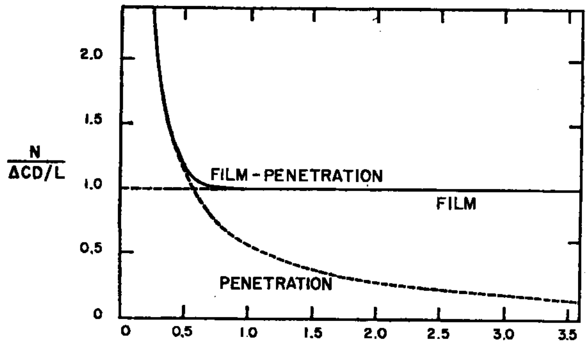

Fig. 1. Point transfer rates as a function of time.

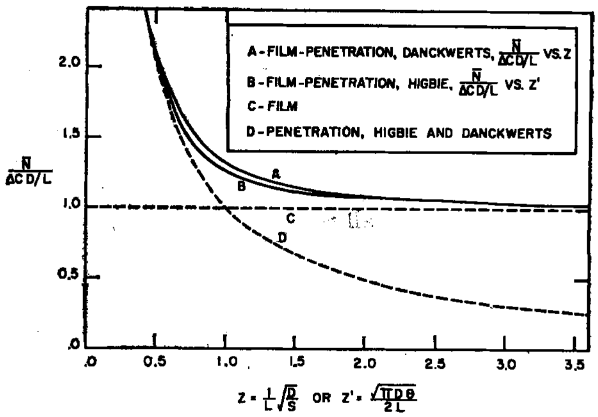

Fig. 2. Mean transfer rates as a function of time.

old, $Z' > 1.0$

$$\bar{N} = \Delta C \frac{D}{L}\left[1 + \frac{1}{3}\frac{L^2}{D\theta'}\right] \qquad (16b)$$

Danckwerts Distribution

young, $Z < 1.0$

$$\bar{N} = \Delta C\sqrt{DS} \cdot \left[1 + 2\exp\left\{-2L\sqrt{\frac{S}{D}}\right\}\right] \qquad (17a)$$

old, $Z > 1.0$

$$\bar{N} = \Delta C \frac{D}{L}\left[1 + \frac{1}{3}\frac{SL^2}{D}\right] \qquad (17b)$$

where

$$Z' = \frac{\sqrt{\pi}}{2}\frac{\sqrt{D\theta'}}{L} \qquad (18)$$

$$Z = \frac{1}{L}\sqrt{\frac{D}{S}} \qquad (19)$$

Since the variables Z' and Z are ratios of film coefficients to penetration coefficients, the equations can be written in terms of D/L and Z' or Z. Figure 2 shows the equations plotted with Z' as the abscissa for Equation (16) and Z as the abscissa for Equation (17). It is clear that the two functions are very closely related, and in fact it would be difficult to distinguish between them experimentally. When Z and Z' are 1.0, the film- and penetration-theory rates are the same and it can be seen that for Z and $Z' < 1$ the film-penetration theory rapidly approaches the penetration theory but for Z and $Z' > 1$ it converges rapidly to the film theory.

It is interesting to note that a five- or tenfold variation in Z or Z' is enough to change a surface from what is essentially a film to one which has the characteristics of a renewed surface. Consequently, a surface defined in the hydrodynamic sense (L and θ' or S fixed) may change from a film mechanism at low Schmidt numbers to a penetration mechanism at high Schmidt numbers.

Similar conclusions may be drawn regarding other types of distribution functions. In general, for surfaces in which old elements predominate, the film theory is valid and the transfer rate varies with the first power of D but, if young elements predominate, the penetration theory applies and the rate varies with the one-half power of D.

The previous discussion applies equally to heat transfer as well as mass transfer and all the foregoing equations have their heat transfer analogues.

The analogue of Equation (13) is

$$\bar{q} = 2\Delta t\sqrt{k\rho c_p S} \cdot \left[1 + 2\sum_{n=1}^{\infty}\exp\left\{-2nL\sqrt{\frac{S}{\alpha}}\right\}\right] \qquad (20a)$$

$$\bar{q} = \Delta t\frac{k}{L}\left[1 + 2 \cdot \sum_{n=1}^{\infty}\frac{1}{1 + n^2\pi^2\frac{\alpha}{SL^2}}\right] \qquad (20b)$$

and the other equations can be written in a similar fashion.

TRANSFER FROM A SOLID TO A LIQUID

An example of the application of these equations is the transfer from a solid to a fluid in turbulent flow. Mass transfer terminology will be employed in this discussion. The common assumption of a laminar film at the solid surface leads to the conclusion that at low values of the diffusivity (high Schmidt numbers) the over-all transfer rate is proportional to the first power of the diffusivity, since the molecular transfer is controlling. Other authors have pointed out that this is inconsistent with experiment, and Lin, Moulton, and Putnam (*9*) as well as Diessler (*4*) have introduced an eddy diffusivity into the film region to make analogy calculations fit the data at high Schmidt numbers. Hanratty (*5*) has also shown that concentration profiles in this region are fitted better by the penetration theory than by the film theory.

The procedure to be followed here is to apply the film-penetration model near the wall in place of the other models in order to determine whether the characteristic shift from a film model to a penetration model is consistent with available data.

It is assumed that the distance L in the model corresponds to a fixed value of y^+, and L is given by

$$L = \frac{L^+ d}{\sqrt{f/2}N_{Re}} \qquad (21)$$

Since θ' and S must be fixed entirely by the hydrodynamics, for smooth tubes with fully developed flow and constant physical properties they must be given by

$$S = \frac{V}{d}f_1(N_{Re}) \qquad (22)$$

$$\theta' = \frac{d}{V}f_2(N_{Re}) \qquad (23)$$

Figure (2) shows that use of either of these variables will give similar results, and so one considers only the Danckwerts distribution function and assumes a power relationship for $f_1(N_{Re})$,

$$S = a\frac{V}{d}(N_{Re})^m \qquad (24)$$

Equations (17) and (19) with (21) and (24) yield for the transfer coefficient from the interface to point L,

$Z < 1.0$

$$\frac{k_f}{V} = \sqrt{\frac{a}{N_{Sc}}(N_{Re})^{m-1}} \cdot \left[1 + 2\exp\left\{-2L^+\sqrt{\frac{aN_{Sc}(N_{Re})^{m-1}}{f/2}}\right\}\right] \qquad (25a)$$

$Z > 1.0$

$$\frac{k_f}{V} = \frac{\sqrt{f/2}}{L^+ N_{Sc}} \cdot \left[1 + \frac{aL^{+2}N_{Sc}(N_{Re})^{m-1}}{3f/2}\right] \qquad (25b)$$

where

$$k_f = \frac{\bar{N}}{C_i - C_L} \tag{26}$$

$$Z = \frac{1}{L^+}\sqrt{\frac{f/2}{aN_{Sc}(N_{Re})^{m-1}}} \tag{27}$$

Except at high Schmidt numbers $1/k_f$ is only part of the total resistance to transfer, and the measured transfer coefficients contain the effects of the resistance in all regions of the fluid. It is possible from the work of Lin, Moulton, and Putnam (*9*) to obtain an equation for k_f which, in combination with analogy calculations in the buffer and turbulent regions, is consistent with experimental data over wide ranges of the Schmidt and Reynolds numbers. This equation is of the form

$$\frac{k_f}{V\sqrt{f/2}} = f_3(N_{Sc}) \tag{28}$$

and is shown in Figure (3) for k_f defined over the region $y^+ = 0$ to $y^+ = 5$.

When Equation (25) is in the form above and the equation for the friction factor,

$$\frac{f}{2} = 0.023(N_{Re})^{-0.2} \tag{29}$$

is used,

$Z < 1.0$

$$\frac{k_f}{V\sqrt{f/2}} = \sqrt{\frac{a}{0.023}\frac{1}{N_{Sc}}}\,(N_{Re})^{m-0.8/2} \cdot\left[1 + 2\exp\left\{-2L^+\sqrt{\frac{a}{0.023}N_{Sc}}\cdot(N_{Re})^{m-0.8/2}\right\}\right] \tag{30a}$$

$Z > 1.0$

$$\frac{k_f}{V\sqrt{f/2}} = \frac{1}{L^+N_{Sc}}\left[1 + \frac{aL^{+2}}{0.069}N_{Sc}(N_{Re})^{m-0.8}\right] \tag{30b}$$

If there is to be no Reynolds-number–Schmidt-number interaction in these equations, the constant m must be set equal to 0.8. Equation (30) is then of the same form as Equation (28),

$Z < 1.0$, high N_{Sc}

$$\frac{k_f}{V\sqrt{f/2}} = \sqrt{\frac{a}{0.023}}\frac{1}{\sqrt{N_{Sc}}}\left[1 + 2\exp\left\{-2L^+\sqrt{\frac{a}{0.023}}\sqrt{N_{Sc}}\right\}\right] \tag{31a}$$

$Z > 1.0$, low N_{Sc}

$$\frac{k_f}{V\sqrt{f/2}} = \frac{1}{L^+N_{Sc}}\left[1 + \frac{aL^{+2}}{0.069}N_{Sc}\right] \tag{31b}$$

and

$$Z = \frac{1}{L^+}\sqrt{\frac{0.023}{a}}\frac{1}{\sqrt{N_{Sc}}} \tag{32}$$

Setting L^+ equal to 5 and adjusting the parameter a gives the results shown in Figure 3, where the curves for three values of a are presented.

The curve of Lin, Moulton, and Putnam (*9*), which has a slope of −1 at low Schmidt numbers, changes to a slope of −⅔ at high Schmidt numbers. The present model changes from a slope of −1 at low Schmidt numbers to a slope of −½ at high Schmidt numbers. This change in slope is a characteristic of the model, and variation of the constants in Equation (25) merely causes a shift in the transition points. As a increases in Figure 3, for example, the transition region shifts to higher Schmidt numbers but it follows from Figure 2 that the transition is essentially completed over the same hundredfold variation in Schmidt number in all cases.

The data of Linton and Sherwood (*10*) and Lin, et al. (*8*) over a range of N_{Sc} from 300 to 3,000 can be compared directly with these curves, but to avoid confusion these data are not shown. Although the data scatter somewhat they are more in agreement with the line of slope −⅔ than −½.

The data of Johnson and Huang (*7*), however, obtained in a stirred tank over a range of Schmidt number from 200 to 15,000, indicate a −½ power on Schmidt number which is consistent with the present model, since all these data are at high Schmidt numbers. This disagrees with the slope of −⅔ in conduits at high Schmidt numbers and it may be that transfer in agitated vessels differs basically from transfer in conduits. In terms of the present model the difference could be due to the presence of a velocity gradient in the fluid near the wall of a conduit, which may not be present in the stirred tank. Since the model neglects this gradient it would check the stirred-tank data, but not the conduit data. Clearly, more mass transfer data are needed before any positive conclusions can be drawn, especially at high Schmidt numbers in conduits and low Schmidt numbers in stirred tanks.

If the penetration model does hold at high Schmidt numbers, it cannot be true at low Schmidt numbers. This is shown in Figure 3, where the line of slope −½ is extended to low values of N_{Sc} and is off at least a factor of 10. In the same manner the line of slope −1 corresponding to the film theory is off a factor of 10 at high Schmidt numbers so that neither model alone can fit the data over the entire range.

However, the film-penetration model, which contains the foregoing models as limits, is fairly consistent with available rate data in conduits over the full range, and since it reduces to the penetration theory at high Schmidt numbers, it is also consistent with concentration profiles (*5*) as well as with Johnson and Huang's (*7*) data.

Both the eddy diffusivity and film-penetration models are crude representa-

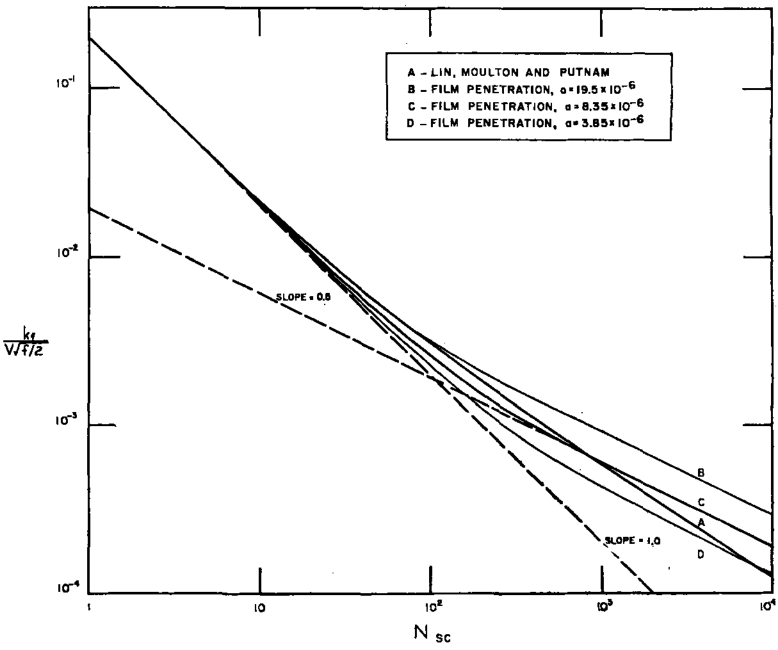

Fig. 3. Transfer coefficients from wall to $y^+ = 5$ as a function of Schmidt number.

tions of a very complicated process; however, the latter model gives a clearer (and possibly more misleading) picture of the exchange mechanism between the wall and the fluid. In this case one pictures masses of fluid moving from the buffer layer to the wall and displacing the fluid previously near the wall. This renewal takes place at a rate described by

$$S = a \frac{V}{d} (N_{Re})^{0.8} \qquad (33)$$

if it is assumed that there is no $N_{Re} - N_{Sc}$ interaction in k_f. Thus this assumption makes the fractional rate of surface renewal vary with velocity in the same manner as the shear stress. Rough calculations indicate that the momentum transferred by this interchange is a small fraction of the total momentum transfer.

For water at 70°F. in a 1-in. tube S varies with N_{Re} as shown in Table 1. Since Figure 2 shows that the transfer equations are essentially the same whether the renewal is assumed to be random or to occur at a fixed age of the surface elements, it can be considered that θ' is the average life of the elements at the surface, and from Equations (18), (19), and (33) one can write

$$\theta' = \frac{4}{\pi}\frac{1}{S} = \frac{4}{\pi}\frac{d}{aV}(N_{Re})^{-0.8} \qquad (34)$$

Some values of θ' are given in Table 1.

It is interesting to examine the assumption made earlier that C_L is constant over the life of a surface element. Since the form of the distribution function has been shown to be immaterial, it is simplest to show that C_L does not vary significantly over the average life of a surface element, θ'. This variation can take place only if the elements are moving along the wall (if oscillations are neglected at a point).

A material balance along a differential length of tube yields

$$\frac{\pi d^2}{4} V\, d\bar{C} = k_L(C_i - \bar{C})\pi\, d\, dx \qquad (35)$$

and integrating over the distance between two successive mixings, x', with C_i constant yields

$$\frac{\bar{C}_2 - \bar{C}_1}{C_i - \bar{C}_2} = \left(\exp\left\{\frac{4k_L x'}{dV}\right\} - 1\right) \qquad (36)$$

The change in concentration at $y = L$ is approximately related to the bulk-concentration change by

$$C_{L_2} - C_{L_1} = \frac{k_L}{k_f}(\bar{C}_2 - \bar{C}_1) \qquad (37)$$

and combining Equations (36) and (37) gives

$$\frac{C_{L_2} - C_{L_1}}{C_i - \bar{C}_2} = \frac{k_L}{k_f}\left(\exp\left\{\frac{4k_L x'}{dV}\right\} - 1\right) \qquad (38)$$

A sufficient condition for C_L to be treated as constant is that the group on the left approach zero. If the average velocity of the fluid in the region from $y^+ = 0$ to $y^+ = 5$ is taken as the velocity at $y^+ = 2.5$ and since $x' = \bar{u}_f\theta'$,

$$x' = 2.5V\sqrt{f/2}\,\theta' \qquad (39)$$

and using the Colburn equation for k_L and Equation (34) for θ' results in the exponent in Equation (38) being

$$\frac{4k_L x'}{dV} = \frac{40a}{\pi}\left(\frac{f}{2}\right)^{3/2}(N_{Sc})^{-2/3}(N_{Re})^{-0.8} \qquad (40)$$

In the worst case this exponent is less than 10^{-7} and since k_L/k_f is less than 1, Equation (38) shows that any variation of C_L due to the displacement of a surface element along the wall is indeed negligible.

TABLE 1

RATE OF SURFACE RENEWAL AND AVERAGE LIFE OF SURFACE ELEMENTS
Water, 70°F., 1-in. tube, $a = 8.35 \times 10^{-6}$

N_{Re}	S, sec.$^{-1}$	θ', sec.
1×10^4	0.21	6.2
5×10^4	3.6	0.36
1×10^5	13.0	0.10

CONCLUSIONS

The transfer to young elements of a surface follows the penetration theory, since the penetration has not reached the outer edge of the elements, and the transfer into the old elements follows the film theory, for the steady state has been attained in these elements.

At low Schmidt numbers the steady gradient is set up very rapidly in any new surface element so that unless the rate of renewal is high enough to remove a large fraction of the surface elements before they are penetrated, most of the surface is, in the foregoing sense, old. Steady state transfer then takes place through what is essentially a film and the effect of surface renewal is negligible.

As the Schmidt number increases, the time necessary to set up the steady gradient increases rapidly, and even low rates of surface renewal are sufficient to keep most of the elements from being penetrated. The transfer then follows the penetration theory and the transfer rate is a function of the rate of surface renewal.

When conditions are such that a surface contains appreciable quantities of young and old elements, as well as middle-aged ones, the transfer characteristics are intermediate between the film and penetration types.

NOTATION

a = constant
A = area
A' = total area
c_p = heat capacity
C = concentration
$\bar{C}$ = bulk concentration
C_L = concentration at $y = L$
C_i = interfacial concentration
ΔC = $C_i - C_L$
d = diameter
D = diffusivity
f = Fanning friction factor
f_1, f_2, f_3 = functional symbols
k = thermal conductivity
k_f = film mass transfer coefficient
k_L = mass transfer coefficient
L = thickness of region in which molecular transfer is controlling
L^+ = value of y^+ at $y = L$
m = constant
N = point mass transfer rate
$\bar{N}$ = mean mass transfer rate
N_{Re} = Reynolds number, $dV\rho/\mu$
N_{Sc} = Schmidt number, $\mu/\rho D$
$\bar{q}$ = mean heat transfer rate per unit area
S = fractional rate of surface renewal
Δt = interfacial temperature minus temperature at $y = L$
V = mean velocity
x = axial distance
x' = distance between two successive mixings
y = distance from interface
y^+ = $(L/d)\sqrt{f/2}N_{Re}$
Z = $(1/L)\sqrt{D/S}$
Z' = $(1/2L)\sqrt{\pi D\theta'}$

Greek Letters

α = thermal diffusivity
θ = time
θ' = contact time
μ = viscosity
ρ = density
$\psi(\theta)$ = surface-age distribution function

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